SOME MORE RESULTS ON RATES OF CONVERGENCE IN THE LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES(1)

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1. **Introduction and summary.** Let X_N for $N=1, 2, \ldots$ be an independent sequence of random variables with finite first absolute moments; let $a_{N,k}$ for $N, k=1, 2, \ldots$ be real numbers; let

$$S_N = \sum_{k=1}^{\infty} a_{N,k}(X_k - EX_k); \tag{1.1}$$

and let

$$A_N = \frac{1}{N} \sum_{k=1}^{N} (X_k - EX_k). \tag{1.2}$$

Both classical and modern results deal with the convergence of the sequences A_N and $P\{|A_N| > \varepsilon\}$ to zero, this convergence being of obvious practical interest. In the past few years sums of the form (1.1) have received attention, partly due to the natural tendency of mathematicians to generalize known results, and partly because sums of the form

$$\frac{1}{N} \sum_{k=1}^{N} \left[\sum_{j=0}^{\infty} a_j(X_{k-j} - EX_{k-j}) \right]$$

are of interest in certain practical statistical and probabilistic problems. We have not attempted to provide a complete listing of the work done in this area. [1] through [11] contain several different types of current work, and provide a fairly large, though by no means complete, bibliography for this area.

The results given here extend and sharpen the results of [6] giving rates of convergence of $P\{|S_N| > \varepsilon\}$ to zero. There are five theorems in [6], each involving as a hypothesis some condition closely related to the existence of $E|X_k - EX_k|^t$. In the next section we extend and sharpen some of these theorems by considering separately the cases $t \leqq 2$ and $t > 2$; in addition, we provide a theorem giving series convergence under weaker moment conditions than were used in [6] but under

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stronger assumptions on the $a_{N,k}$'s. In §3 we show that Theorem 5 of [6] is actually a corollary to Theorem 4 of [6] and investigate the sharpness of Theorem 4. §4 contains some miscellaneous concluding remarks.

2. **Extensions of previous results.** Throughout this paper C will denote various positive constants whose exact numerical values do not matter. Using this notation inequalities such as $1+C \leqq C$ are valid.

Let X_N, $a_{N,k}$, and S_N be defined as in §1. Suppose t and t' are constants and $\{\alpha_N\}$, $\{\gamma_N\}$, and $\{\rho_N\}$ are sequences of positive numbers such that

$$\sum_k |a_{N,k}|^{t'} \leqq \alpha_N, \tag{2.1}$$

$$\sum_k a_{N,k}^2 \leqq \gamma_N, \tag{2.2}$$

and

$$\sum_k |a_{N,k}|^t \leqq \rho_N. \tag{2.3}$$

For $y \geqq 0$ define

$$F(y) = \sup_k P\{|X_k - EX_k| \geqq y\}. \tag{2.4}$$

We will prove the following theorems:

THEOREM 1a. *If $1 \leqq t \leqq 2$ and $E|X_k - EX_k|^t \leqq M < \infty$ for all k, then for every $\varepsilon > 0$*

$$P\{|S_N| > \varepsilon\} \leqq O(\rho_N). \tag{2.5}$$

THEOREM 1b. *If $1 \leqq t' < t \leqq 2$, if $y^t F(y) \leqq M < \infty$ for all $y > 0$, and if there exists a λ_0 such that $\alpha_N \rho_N^{\lambda_0} < M' < \infty$ for all N, then* (2.5) *holds for every $\varepsilon > 0$.*

THEOREM 1c. *If $t > 2$, if $y^t F(y) \leqq M < \infty$ for all $y > 0$, and if there exists a $\nu_0 > 0$ such that $\gamma_N^{\nu_0} \leqq O(\rho_N)$, then* (2.5) *holds for every $\varepsilon > 0$.*

THEOREM 2a. *If $1 \leqq t \leqq 2$, if $E|X_k - EX_k|^t \leqq M < \infty$ for all k, if $y^t F(y) \to 0$ as $y \to \infty$, and if $\rho_N \to 0$ as $N \to \infty$, then for every $\varepsilon > 0$*

$$P\{|S_N| > \varepsilon\} = o(\rho_N). \tag{2.6}$$

THEOREM 2b. *If $1 \leqq t' < t \leqq 2$, if $y^t F(y) \to 0$ as $y \to \infty$, if $\rho_N \to 0$ as $N \to \infty$, and if there exists a λ_0 such that $\alpha_N \rho_N^{\lambda_0} \leqq M < \infty$ for all N, then* (2.6) *holds for every $\varepsilon > 0$.*

THEOREM 2c. *If $t > 2$, if $y^t F(y) \to 0$ as $y \to \infty$, if $\rho_N \to 0$ as $N \to \infty$, and if there exists a $\nu_0 > 0$ such that $\gamma_N^{\nu_0} \leqq O(\rho_N)$, then* (2.6) *holds for every $\varepsilon > 0$.*

In Theorems 3, 4, and 5 we assume that α_N, γ_N, and ρ_N are of the form CN^{α}, $CN^{-\gamma}$, and $CN^{-\rho}$ respectively. In addition, for Theorems 3 and 5 we assume there exist constants C and β such that

$$\max_k |a_{N,k}| \leqq CN^{-\beta}. \tag{2.7}$$

For $s=t$ and $s=2$,

$$\left[\max_k |a_{N,k}|\right]^s \leqq \sum_k |a_{N,k}|^s$$

so we may assume

$$\beta t \geqq \rho \quad \text{and} \quad 2\beta \geqq \gamma. \tag{2.8}$$

For $t>2$

$$\sum_k |a_{N,k}|^t \leqq \left[\max_k |a_{N,k}|\right]^{t-2} \sum_k a_{N,k}^2$$

so if $t>2$ we may assume

$$\rho \geqq \beta(t-2)+\gamma. \tag{2.9}$$

Define for $N, M=1, 2, \ldots$

$$\nu_{N,M} = \text{cardinality}\,\{k : |a_{N,k}| \geqq M^{-1}\}. \tag{2.10}$$

THEOREM 3. *If $t>2$, $\gamma>0$, and F satisfies*

$$\lim_{y\to\infty} F(y) = 0 \quad \textit{and} \quad \int_0^\infty y^t |dF(y)| < \infty, \tag{2.11}$$

then for every $\varepsilon>0$

$$\sum_N N^{\beta(t-2)+\gamma-1} P\{|S_N| > \varepsilon\} < \infty. \tag{2.12}$$

THEOREM 4. *If $t\geqq 1$; $\rho>0$; $\gamma>0$; and there exists a nonnegative and nonincreasing real valued function G satisfying* (2.11), *$G(x)\geqq F(x)$ for all x, and*

$$\sup_{x\geqq 1} \sup_{y\geqq x} \frac{y^t F(y)}{x^t G(x)} = \eta < \infty, \tag{2.13}$$

then for every $\varepsilon>0$

$$\sum_N N^{\rho-1} P\{|S_N| > \varepsilon\} < \infty. \tag{2.14}$$

THEOREM 5. *If $t\geqq 1$, $\rho>0$, $\gamma>0$, F satisfies* (2.11), *and either*

(a) *$\nu_{N,M}\leqq CM^t N^{-(\rho-1)} g(N)$ with $\sum_N g(N)<\infty$, or*

(b) *$\nu_{N,M}\leqq CM^\mu N^{-\sigma}$ with $\sigma\geqq\mu\beta$, and either $\rho\neq\sigma$ or $t\neq\mu$,*

then (2.14) *holds for every $\varepsilon>0$.*

Proofs. Throughout these proofs summations will be taken over those values of k for which $a_{N,k}\neq 0$. Initially we prove Theorems 1a, 1b, and 1c under the assumption that $\rho_N\to 0$ as $N\to\infty$; the other cases will be taken care of at the end of this section.

The proofs begin as those in [6] where the following inequality was proved:

$$P\{|S_N| > 3\varepsilon\}$$

$$\leqq \sum_k P\{|a_{N,k}(X_k - EX_k)| > \varepsilon\} \tag{2.15}$$

$$+ \sum_{j \neq k} P\{|a_{N,k}(X_k - EX_k)| > \delta_N\} P\{|a_{N,j}(X_j - EX_j)| > \delta_N\} \tag{2.16}$$

$$+ P\left\{\left|\sum_k a_{N,k} EY_{N,k}\right| > \varepsilon\right\} \tag{2.17}$$

$$+ P\left\{\left|\sum_k a_{N,k}(Y_{N,k} - EY_{N,k})\right| > \varepsilon\right\}, \tag{2.18}$$

where δ_N is a sequence of positive numbers to be chosen later and

$$\begin{aligned} Y_{N,k} &= X_k - EX_k \quad \text{if } |a_{N,k}(X_k - EX_k)| \leqq \delta_N \\ &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

The proof of each theorem proceeds by showing that expressions (2.15)–(2.18) tend to zero at the rate specified in the theorem.

Expression (2.15). For Theorems 1a, 1b, 1c, 2a, 2b, and 2c, we observe that (2.15) is bounded by

$$\sum_k \frac{|a_{N,k}|^t}{\varepsilon^t} \sup_{y \geqq \varepsilon/|a_{N,k}|} [y^t P\{|X_k - EX_k| \geqq y\}] \leqq C\rho_N \sup_{y^t \geqq C\rho_N^{-1}} [y^t F(y)].$$

The hypotheses of Theorems 1a, 1b, and 1c insure that

$$\sup_k \sup_{y \geqq \varepsilon/|a_{N,k}|} y^t P\{|X_k - EX_k| \geqq y\} < \infty$$

so that (2.15) is $O(\rho_N)$. The hypotheses of Theorems 2a, 2b, and 2c insure that

$$\sup_{y^t \geqq C\rho_N^{-1}} [y^t F(y)] \to 0 \quad \text{as } N \to \infty.$$

Hence (2.15) is $o(\rho_N)$ under the hypotheses of these theorems.

For Theorem 3 we note that

$$\begin{aligned} \sum_N N^{\beta(t-2)+\gamma-1} \sum_k & P\{|a_{N,k}(X_k - EX_k)| > \varepsilon\} \\ & \leqq \sum_N N^{\beta(t-2)+\gamma-1} \sum_k F(\varepsilon/|a_{N,k}|) \\ & \leqq \sum_{M=1}^{\infty} F(M-1) \sum_{\{(N,k)\,:\, M-1 < \varepsilon/|a_{N,k}| \leqq M\}} N^{\beta(t-2)+\gamma-1} \qquad (2.19) \\ & = \sum_{M=1}^{\infty} [F(M-1) - F(M)] \sum_{N=1}^{\infty} N^{\beta(t-2)+\gamma-1} \nu_{N,M}(\varepsilon), \end{aligned}$$

where $\nu_{N,M}(\varepsilon) = \text{card}\,\{k : \varepsilon/|a_{N,k}| \leqq M\}$. Since $(\varepsilon^2/M^2)\nu_{N,M}(\varepsilon) \leqq CN^{-\gamma}$ and $|a_{N,k}| \leqq CN^{-\beta}$ for all k, it is clear that $\nu_{N,M}(\varepsilon) \leqq CM^2N^{-\gamma}$ and $\nu_{N,M}(\varepsilon) = 0$ unless $N \leqq CM^{1/\beta}$.

Thus (2.19) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^2\sum_{N=1}^{[CM^{1/\beta}]}N^{\beta(t-2)-1}$$
$$\leqq C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^t \leqq C+C\int_0^{\infty}x^t|dF(x)| < \infty.$$

Under the assumptions of Theorem 4 the finiteness of

$$\sum_N N^{\rho-1}\sum_k P\{|a_{N,k}(X_k-EX_k)| > \varepsilon\} \tag{2.20}$$

is proved on pages 351 and 352 of [6] starting at (20).

An argument similar to that given for Theorem 3 shows that (2.20) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]\sum_{N=1}^{[CM^{1/\beta}]}N^{\rho-1}\nu_{N,M}(\varepsilon). \tag{2.21}$$

It is clear that there exists a positive integer r such that $\nu_{N,M}(\varepsilon)\leqq \nu_{N,rM}$, and so under assumption (a) of Theorem 5 expression (2.21) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^t\sum_{N=1}^{[CM^{1/\beta}]}g(N) \leqq C+C\int_0^{\infty}x^t|dF(x)| < \infty.$$

Under assumption (b) of Theorem 5 expression (2.21) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^{\mu}\sum_{N=1}^{[CM^{1/\beta}]}N^{\rho-1-\sigma}. \tag{2.22}$$

If $\rho\neq\sigma$ and $\sigma\geqq\mu\beta$, then $\sum_{N=1}^{[CM^{1/\beta}]}N^{\rho-1-\sigma}\leqq CM^{(\rho-\sigma)/\beta}$; from (2.8) we get

$$\frac{\rho-(\sigma-\beta\mu)}{\beta}\leqq\frac{\rho}{\beta}\leqq t;$$

thus expression (2.22) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^t \leqq C+C\int_0^{\infty}x^t|dF(x)| < \infty.$$

If $\rho=\sigma$, then by hypothesis $\mu\neq t$ and this implies that $\mu<t$. Thus (2.22) is bounded by

$$C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^{\mu}\sum_{N=1}^{[CM^{1/\beta}]}N^{-1}$$
$$\leqq C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^{\mu}\log(M+1)$$
$$\leqq C\sum_{M=1}^{\infty}[F(M-1)-F(M)]M^t \leqq C+C\int_0^{\infty}x^t|dF(x)| < \infty.$$

EXPRESSION (2.16). Expression (2.16) is bounded by

$$\left[\sum_k |a_{N,k}|^t \delta_N^{-t}\left(\left(\frac{\delta_N}{|a_{N,k}|}\right)^t P\left\{|X_k-EX_k| > \frac{\delta_N}{|a_{N,k}|}\right\}\right)\right]^2.$$

The hypotheses of all the theorems insure that $y^tF(y) \leqq M < \infty$ for all $y > 0$, so that (2.16) is bounded by $C\rho_N^2\delta_N^{-2t}$. In order for (2.16) to tend to zero at the appropriate rates it would suffice to show that there exists a $\tau > 0$ such that

(2.23) $$\rho_N^{1-\tau}\delta_N^{-2t} \to 0 \quad \text{as } N \to \infty.$$

The δ_N's will be chosen later in the proof and will satisfy (2.23).

EXPRESSION (2.17). For all the theorems in question it suffices to show that

$$\sum_k a_{N,k}EY_{N,k} \to 0 \quad \text{as } N \to \infty.$$

For Theorems 1a, 2a, 3, 4, and 5 we observe that

$$\begin{aligned}\left|\sum_k a_{N,k}EY_{N,k}\right| &\leqq \sum_k |a_{N,k}| \left|\int_{(|a_{N,k}(X_k-EX_k)| \leqq \delta_N)} (X_k-EX_k)\,dP\right| \\ &= \sum_k |a_{N,k}| \left|\int_{(|a_{N,k}(X_k-EX_k)| > \delta_N)} (X_k-EX_k)\,dP\right| \\ &\leqq \sum_k |a_{N,k}| \int_{(|a_{N,k}(X_k-EX_k)| > \delta_N)} |X_k-EX_k|\,dP \\ &\leqq \delta_N^{-(t-1)} \sum_k |a_{N,k}|^t E|X_k-EX_k|^t \leqq C\delta_N^{-(t-1)}\rho_N.\end{aligned}$$

So Expression (2.17) tends to zero at the rates specified if

(2.24) $$\delta_N^{-(t-1)}\rho_N \to 0 \quad \text{as } N \to \infty.$$

We will choose $\{\delta_N\}$ later so as to satisfy (2.24).

For Theorems 1b, 1c, 2b, and 2c an argument similar to the above gives

$$\left|\sum_k a_{N,k}EY_{N,k}\right| \leqq \delta_N^{-(t-\lambda-1)} \sum_k |a_{N,k}|^{t-\lambda} E|X_k-EX_k|^{t-\lambda},$$

where λ satisfies $0 < \lambda < t-1$ and will be chosen later. The hypotheses of the theorems under consideration insure $y^tF(y) \leqq M < \infty$ for all $y > 0$, which implies that for a fixed λ satisfying $0 < \lambda < t-1$ there exists an M' with

$$E|X_k-EX_k|^{t-\lambda} \leqq M' < \infty \quad \text{for all } k.$$

If $t > 2$, by applying Hölder's inequality and requiring

(2.25) $$0 < \lambda < t-2,$$

we see that

$$\begin{aligned}\sum_k |a_{N,k}|^{t-\lambda} &= \sum_k |a_{N,k}|^{2\lambda/(t-2)}|a_{N,k}|^{t(t-2-\lambda)/(t-2)}\\ &\leqq \Big(\sum_k a_{N,k}^2\Big)^{\lambda/(t-2)}\Big(\sum_k |a_{N,k}|^t\Big)^{(t-2-\lambda)/(t-2)}\\ &\leqq \gamma_N^{\lambda/(t-2)}\rho_N^{1-\lambda/(t-2)}\end{aligned}$$

so for Theorems 1c and 2c it suffices to show that

$$\delta_N^{-(t-\lambda-1)}\gamma_N^{\lambda/(t-2)}\rho_N^{1-\lambda/(t-2)} \to 0 \quad \text{as } N\to\infty. \tag{2.26}$$

Later $\{\delta_N\}$ and λ will be chosen so that (2.26) holds. If $1 \leqq t' < t \leqq 2$, we require

$$0 < \lambda < t-t'. \tag{2.27}$$

Hölder's inequality is applied, as before, and we get

$$\begin{aligned}\sum_k |a_{N,k}|^{t-\lambda} &= \sum_k |a_{N,k}|^{t'(\lambda/(t-t'))}|a_{N,k}|^{t(t-t'-\lambda)/(t-t')}\\ &\leqq \Big(\sum_k |a_{N,k}|^{t'}\Big)^{\lambda/(t-t')}\Big(\sum_k |a_{N,k}|^t\Big)^{(t-t'-\lambda)/(t-t')}\\ &\leqq \alpha_N^{\lambda/(t-t')}\rho_N^{1-\lambda/(t-t')}.\end{aligned}$$

Hence for Theorems 1b and 2b it suffices to choose (as we shall do later) $\{\delta_N\}$ and λ so that

$$\delta_N^{-(t-\lambda-1)}\alpha_N^{\lambda/(t-t')}\rho_N^{1-\lambda/(t-t')} \to 0 \quad \text{as } N\to\infty. \tag{2.28}$$

Expression (2.18). For convenience we let $Z_{N,k} = Y_{N,k} - EY_{N,k}$. As on page 354 of [6], we use Markov's inequality, with ν a fixed positive even integer to be chosen later, and obtain

$$\begin{aligned}P\Big\{\Big|\sum_k a_{N,k}Z_{N,k}\Big| > \varepsilon\Big\} &\leqq \frac{1}{\varepsilon^\nu} E\Big(\sum_k a_{N,k}Z_{N,k}\Big)^\nu\\ &\leqq C\sum_1\sum_2\prod_{k=1}^{a+b}|a_{N,\beta_k}|^{m_k}E|Z_{N,\beta_k}|^{m_k}\end{aligned} \tag{2.29}$$

where the first sum is taken over all integers $a, b, m_1, \dots, m_{a+b}$ such that $2 \leqq m_k < t$ for $k = 1, \dots, a$; $t \leqq m_k$ and $2 \leqq m_k$ for $k = a+1, \dots, a+b$; $\sum_{k=1}^{a+b} m_k = \nu$; and distinct sets of integers $\{m_1, \dots, m_{a+b}\}$ appear only once in the sum; in the second sum $\beta_1, \dots, \beta_{a+b}$ are allowed to range over the positive integers. Note that the first sum is over a finite number (depending on ν but not on N) of terms.

We now proceed to complete the proofs of Theorems 1a, 2a, 3, 4, and 5 by showing that (2.29) is $O(\rho_N^{1+\xi})$ for some $\xi > 0$. For these theorems $E|X_k|^t \leqq M < \infty$ for all k which implies that $E|Z_{N,k}|^j \leqq M' < \infty$ for all N, k and for $j = 1, 2, \dots, [t]$ and $j = t$. Hence if $1 \leqq k \leqq a$

$$\begin{aligned}|a_{N,\beta_k}|^{m_k}E|Z_{N,\beta_k}|^{m_k} &\leqq C|a_{N,\beta_k}|^2|a_{N,\beta_k}|^{m_k-2}\\ &\leqq C\gamma_N^{(m_k-2)/2}a_{N,\beta_k}^2,\end{aligned}$$

and if $a+1 \leqq k \leqq a+b$

$$|a_{N,\beta_k}|^{m_k} E|Z_{N,\beta_k}|^{m_k} \leqq |a_{N,\beta_k}|^t E|Z_{N,\beta_k}|^t \Big[\sup_\omega |a_{N,\beta_k} Z_{N,\beta_k}(\omega)|^{m_k - t}\Big]$$
$$\leqq C\delta_N^{m_k - t}|a_{N,\beta_k}|^t.$$

Therefore each term in the first sum of (2.29) is bounded by

$$C\gamma_N^{(m_1+\cdots+m_a)/2}\rho_N^b\delta_N^{m_{a+1}+\cdots+m_{a+b}-bt}. \tag{2.30}$$

In Theorems 1a and 2a we assume $t<2$ so that $a=0$ in (2.29) and (2.30). Thus (2.30) becomes

$$C\rho_N^b\delta_N^{\nu-bt}. \tag{2.31}$$

For Theorems 1a and 2a we choose $\nu=2$ and $\delta_N=\rho_N^{1/3t}$ so that $b=1$. This choice of δ_N satisfies (2.23) and (2.24). With this choice of δ_N and ν expression (2.31) becomes $C\rho_N^{1+(\nu-t)/3t}$. We note that $\nu-t>0$ and so $1+(\nu-t)/3t>1$.

For Theorems 3, 4, and 5 there exists a $\nu_0>0$ such that $\gamma_N^{\nu_0} \leqq O(\rho_N)$. For these theorems we choose $\nu>\max(t, 2\nu_0)$ and $\delta_N=\rho_N^{1/3t}$. This choice of δ_N satisfies (2.23) and (2.24). The proofs of these theorems are completed upon showing that (2.30) is $O(\rho_N^{1+\xi})$ for some $\xi>0$. If $b=0$ then (2.30) becomes $C\gamma_N^{\nu/2}$ which is of the desired form since $\nu>2\nu_0$. For $b>1$ we note that γ_N and $\rho_N \to 0$ as $N\to\infty$, and that $m_{a+1}+\cdots+m_{a+b}-bt \geqq 0$, and thus that (2.30) is of the desired form. If $b=1$ and $a=0$ expression (2.30) becomes $C\rho_N\delta_N^{\nu-t}$ which is of the desired form, and if $b=1$ and $a>0$, recalling that $\delta_N\to 0$ as $N\to\infty$, $m_{a+1}+\cdots+m_{a+b}-bt\geqq 0$, and $\gamma_N^{\nu_0} \leqq O(\rho_N)$, we see that (2.30) is $O(\rho_N^{1+\xi})$ for some $\xi>0$.

The proofs of Theorems 1b, 1c, 2b, and 2c will be completed by showing (2.29) is $o(\rho_N)$ under the assumption that $y^tF(y)\leqq M<\infty$ for all $y>0$. This assumption implies that for a fixed λ with $0<\lambda<t-1$,

$$E|Z_{N,k}|^j \leqq M' < \infty \quad \text{for all } N \text{ and } k \text{ with } j=2,\ldots,t_0$$

and $j=t-\lambda$, where t_0 is the largest integer with $t_0<t$. As in the above, if $1\leqq k\leqq a$

$$|a_{N,\beta_k}|^{m_k}E|Z_{N,\beta_k}|^{m_k} \leqq C\gamma_N^{(m_k-2)/2}a_{N,\beta_k}^2.$$

If $a+1\leqq k\leqq a+b$

$$|a_{N,\beta_k}|^{m_k}E|Z_{N,\beta_k}|^{m_k} \leqq |a_{N,\beta_k}|^{t-\lambda}E|Z_{N,\beta_k}|^{t-\lambda}\Big[\sup_\omega |a_{N,\beta_k}Z_{N,\beta_k}|^{m_k-t+\lambda}\Big]$$
$$\leqq C\delta_N^{m_k-t+\lambda}|a_{N,\beta_k}|^{t-\lambda}.$$

So each term in the first sum of (2.29) is bounded by

$$C\gamma_N^{(m_1+\cdots+m_a)/2}\,\delta_N^{m_{a+1}+\cdots+m_{a+b}-b(t-\lambda)}\Big(\sum_k |a_{N,k}|^{t-\lambda}\Big)^b. \tag{2.32}$$

For Theorems 1c and 2c where $t>2$ it has been shown earlier that

$$\sum_k |a_{N,k}|^{t-\lambda} \leqq \gamma_N^{\lambda/(t-2)}\rho_N^{1-\lambda/(t-2)}$$

if $0<\lambda<t-2$. Hence (2.32) is bounded by

$$C\gamma_N^{(m_1+\cdots+m_a)/2+b\lambda/(t-2)}\rho_N^{b(1-\lambda/(t-2))}\,\delta_N^{m_{a+1}+\cdots+m_{a+b}-b(t-\lambda)}. \tag{2.33}$$

Now λ, ν, and δ_N are chosen as follows:

$$0 < 2\lambda < t-2$$
$$\nu > \max(2t\nu_0, 3t)$$
$$\delta_N = \max(\rho_N^{1/3t}, \gamma_N^{1/2t}).$$

This choice of δ_N satisfies (2.23). Since $\delta_N \to 0$ as $N \to \infty$ (2.26) is satisfied if

$$\delta_N^{-t}\gamma_N^{\lambda/(t-2)}\rho_N^{1-\lambda/(t-2)} \to 0 \quad \text{as } N \to \infty. \tag{2.34}$$

But from its definition

$$\delta_N^{-t} \leqq (\gamma_N^{-(\lambda/(t-2))/2t}\rho_N^{-(1-\lambda/(t-2))/3t})^t$$

so (2.34) is obtained. For the two theorems in question it only remains to show (2.33) is $o(\rho_N)$. If $b=0$ expression (2.33) becomes $C\gamma_N^{\nu/2}$ which is $o(\rho_N)$ because $\nu>2\nu_0$. For $b>1$, expression (2.33) is $o(\rho_N)$ since $\lambda/t-2<1/2$, γ_N and $\rho_N \to 0$ as $N \to \infty$, and the exponents on γ_N and ρ_N in (2.33) are nonnegative. If $b=1$, then (2.33) is bounded by

$$C\delta_N^{\nu}(\gamma_N^{(m_1+\cdots+m_a)/2}\,\delta_N^{-(m_1+\cdots+m_a)})(\delta_N^{-t}\rho_N^{1-\lambda/(t-2)}\gamma_N^{\lambda/(t-2)}). \tag{2.35}$$

Now $\delta_N^{\nu}=o(\rho_N)$,

$$\gamma_N^{(m_1+\cdots+m_a)/2}\,\delta_N^{-(m_1+\cdots+m_a)} \leqq \gamma_N^{(m_1+\cdots+m_a)/2-(m_1+\cdots+m_a)/2t} \to 0 \quad \text{as } N \to \infty,$$

and by (2.34)

$$\delta_N^{-t}\rho_N^{1-\lambda/(t-2)}\gamma_N^{\lambda/(t-2)} \to 0 \quad \text{as } N \to \infty.$$

For Theorems 1b and 2b where $1 \leqq t' < t \leqq 2$ it has been shown earlier that

$$\sum_k |a_{N,k}|^{t-\lambda} \leqq \alpha_N^{\lambda/(t-t')}\rho_N^{1-\lambda/(t-t')}$$

if $0<\lambda<t-t'$. We note that for $t \leqq 2$ we have $a=0$ in expression (2.32) which is then bounded by

$$C\delta_N^{\nu-b(t-\lambda)}\alpha_N^{b\lambda/(t-t')}\rho_N^{b(1-\lambda/(t-t'))}. \tag{2.36}$$

For these two theorems λ, ν, and δ_N are chosen as follows:

$$0 < \lambda < 2(t-t')/3(\lambda_0+1)$$
$$\nu > 3t$$
$$\delta_N = \rho_N^{1/3t}.$$

This choice of δ_N satisfies (2.23). Since $\delta_N \to 0$ as $N \to \infty$ condition (2.28) is satisfied if

$$\delta_N^{-t}\alpha_N^{\lambda/(t-t')}\rho_N^{1-\lambda/(t-t')} \to 0 \quad \text{as } N \to \infty. \tag{2.37}$$

By the choice of δ_N expression (2.37) becomes

$$\left(\alpha_N \rho_N^{(2/3-\lambda/(t-t'))(t-t')/\lambda}\right)^{\lambda/(t-t')},$$

and by the choice of λ the exponent on ρ_N is greater than λ_0 so (2.37) holds. It only remains to show (2.36) is $o(\rho_N)$. Expression (2.36) can be rewritten as

$$C\delta_N^{\nu+b\lambda}\left(\delta_N^{-t}\alpha_N^{\lambda/(t-t')}\rho_N^{1-\lambda/(t-t')}\right)^b. \tag{2.38}$$

Applying (2.37) and observing that $\delta_N^{\nu}=o(\rho_N)$ the proofs of these theorems are completed.

Now we relax the assumption that $\rho_N \to 0$ as $N \to \infty$ for Theorems 1a, 1b, and 1c. Define

$$B_k = \{b_N : P\{|S_N| > \varepsilon\} = b_N\rho_N \text{ for } N\text{'s such that } 1/(k+1) < \rho_N \leqq 1/k\}.$$

Now sup B_k is finite for each k and if $\sup_k (\sup B_k) < \infty$ the proof is complete. If not, that is if

$$\sup_k (\sup B_k) = \infty,$$

then there exists a sequence ρ_{N_k} with $\rho_{N_k} \to 0$ as $k \to \infty$ and $P\{|S_{N_k}| > \varepsilon\}/\rho_{N_k} \to \infty$ as $k \to \infty$. This contradiction completes the proof.

3. **Further investigation of Theorem 4 and of condition (2.13).** We continue using the notation of the previous sections, except that we allow $0 < t < 1$; in these cases $E|X_N|$ may not be finite for any N so in these cases we define

$$S_N = \sum_{k=1}^{\infty} a_{N,k}X_k,$$

and

$$F(x) = \sup_k P\{|X_k| \geqq x\}. \tag{3.1}$$

F will always be a function which could have been obtained via (2.4) or (3.1) depending on whether $t \geqq 1$ or $t < 1$ respectively. In the following definitions both $\mathscr{G}$ and G^* correspond to a specific F, but for convenience we will not use an F subscript in either case.

Let

$$\mathscr{G} = \left\{ G \;\middle|\; \begin{array}{l} G : [0, \infty) \to [0, \infty) \text{ and is nonincreasing} \\ G(x) \geqq F(x) \text{ for } 0 \leqq x < \infty \\ G \text{ satisfies (2.11) and (2.13)} \end{array} \right\} \tag{3.2}$$

and define

$$\begin{aligned} G^*(x) &= \operatorname{Max}\left\{1, \sup_{y \geqq 1} y^t F(y)\right\} && 0 \leqq x \leqq 1 \\ &= \frac{1}{x^t} \sup_{y \geqq x} y^t F(y) && 1 < x. \end{aligned} \tag{3.3}$$

The results of this section are summarized in the statements of the following propositions and theorems.

PROPOSITION 1. *If* $\mathscr{G} \neq \varnothing$ *then* $G^* \in \mathscr{G}$. *If* G^* *is finite and satisfies* (2.11) *then* $G^* \in \mathscr{G}$.

THEOREM 6. *If* $t>0$, $\int_0^\infty y^t \log^+ y|dF(y)|<\infty$, *and* $F(y) \to 0$ *as* $y \to \infty$, *then* $\mathscr{G} \neq \varnothing$. *The converse is not true.*

Proposition 1 provides a method for checking to see whether the hypotheses of Theorem 4 are satisfied in a particular case. It also is useful in obtaining some of the main results of this section.

Theorem 6 shows that one can ignore the G of Theorem 4 entirely if $\lim_{y\to\infty} F(y) = 0$ and $\int_0^\infty y^t \log^+ y|dF(y)|<\infty$. It also shows that Theorem 5 of [4] is a corollary to Theorem 4 of [4]—although showing this to be the case was almost as hard as proving Theorem 5 of [4] directly. The following theorem shows that Theorem 6 is tight in the sense that under any reduction in the moment requirement on F, an F can be found such that $\mathscr{G} = \varnothing$.

THEOREM 7. *If* $t>0$ *and* g *is a nonnegative real valued function on* $[0, \infty)$ *such that* $\limsup_{x\to\infty} g(x)=\infty$, *then there exists an* F *such that* $\lim_{x\to\infty} F(x)=0$ *and*

$$\int_0^\infty \frac{x^t \log^+ x}{g(x)} |dF(x)| < \infty$$

but such that $\mathscr{G} = \varnothing$.

Theorems 8 and 9 show that slight weakenings of the hypotheses of Theorem 4 allow $\sum_{N=1}^\infty N^{\rho-1}P\{|S_N|>\varepsilon\}=\infty$, and thus show that Theorem 4 and the result obtained by combining it with Theorem 6 are in a sense sharp.

THEOREM 8. *If* $0<t\leqq 2$, $\rho>0$, *and* F *(corresponding to* G^**) is such that* $\lim_{x\to\infty} F(x) = 0$ *and such that either* $G^*(x)=\infty$ *for some* x *or* $\int_0^\infty x^t|dG^*(x)|=\infty$, *then there exist independent and identically distributed random variables* $X_1, X_2, \ldots$ *and coefficients* $\{a_{N,k}\}$ *such that* $P\{|X_k| \geqq x\} \leqq F(x)$ *for all* $x \geqq 0$,

$$\sum_{k=1}^\infty |a_{N,k}|^t \leqq CN^{-\rho},$$

and

$$\sum_{N=1}^\infty N^{\rho-1}P\{|S_N| \geqq 1\} = \infty.$$

THEOREM 9. *If* $t>0$, $\rho>0$, *and* g *is a nonnegative real-valued function on* $[0, \infty)$ *such that* $\limsup_{x\to\infty} g(x)=\infty$, *then there exists an* F *such that* $\lim_{x\to\infty} F(x)=0$ *and*

$$\int_0^\infty \frac{x^t \log^+ x}{g(x)} |dF(x)| < \infty;$$

independent and identically distributed random variables $X_1, X_2, \ldots$ such that $P\{|X_k| \geqq x\} = F(x)$ for all $x \geqq 0$; and a positive constant γ and coefficients $\{a_{N,k}\}$ such that

$$\sum_{k=1}^{\infty} |a_{N,k}|^2 \leqq CN^{-\gamma}, \qquad \sum_{k=1}^{\infty} |a_{N,k}|^t \leqq CN^{-\rho},$$

and

$$\sum_{k=1}^{\infty} N^{\rho-1} P\{|S_N| \geqq 1\} = \infty.$$

Since for $0 < t \leqq 2$ we have

$$\sum_{k=1}^{\infty} |a_{N,k}|^2 \leqq \sum_{k=1}^{\infty} |a_{N,k}|^t \leqq CN^{-\rho}$$

for large enough N, it follows that for $0 < t \leqq 2$ Theorem 9 is just a corollary to Theorems 7 and 8.

The rest of this section consists of the proofs of Proposition 1; Theorems 6, 7, and 8; and Theorem 9 when $t > 2$.

Proof of Proposition 1. Suppose G^* is finite and satisfies (2.11). It is clearly nonincreasing, and using its definition one quickly sees that $G^*(x) \geqq F(x)$ for $0 \leqq x < \infty$. Now

$$\sup_{x \geqq 1} \sup_{y \geqq x} \frac{y^t F(y)}{x^t G^*(x)} = \max \left\{ \frac{\sup_{y \geqq 1} y^t F(y)}{\max\,[1, \sup_{y \geqq 1} y^t F(y)]}, \sup_{x > 1} 1 \right\}$$
$$= 1 < \infty.$$

Thus $G^* \in \mathscr{G}$.

Suppose $G \in \mathscr{G}$ and $\sup_{x \geqq 1} \sup_{y \geqq x} y^t F(y)/x^t G(x) = \lambda$. If $\lambda \leqq 1$ set $\bar{G} = G$; if $\lambda > 1$ set $\bar{G} = \lambda G$. In either case $\bar{G} \in \mathscr{G}$ and

$$\sup_{x \geqq 1} \sup_{y \geqq x} \frac{y^t F(y)}{x^t \bar{G}(x)} \leqq 1.$$

This implies that $G^*(x)$ is finite for all x, and that $\sup_{y \geqq x} y^t F(y) \leqq x^t \bar{G}(x)$ for all $x > 1$, or equivalently that $G^*(x) \leqq \bar{G}(x)$ for all $x > 1$. Thus

$$\lim_{x \to \infty} G^*(x) \leqq \lim_{x \to \infty} \bar{G}(x) = 0$$

and

$$\int_0^\infty y^t |dG^*(y)| \leqq C + \int_{(1,\infty)} y^t |dG^*(y)|$$
$$\leqq C + \int_{(1,\infty)} y^t |d\bar{G}(y)| < \infty.$$

Thus G^* is finite and satisfies (2.11) so by the first part of the theorem $G^* \in \mathscr{G}$.

Proof of Theorem 6. Suppose $t>0$, $F(y)\to 0$ as $y\to\infty$, and $\int_0^\infty y^t \log^+ y|dF(y)|<\infty$. Define $F^*(x)=1$ for $0\leqq x\leqq 3$, and $F^*(x)=F(N+)$ for $N=3, 4,\ldots$ and $N<x\leqq N+1$. Then $F^*(x)\to 0$ as $x\to\infty$ and F^* can be thought of as the "tail" of a distribution function. In addition $F^*(x)\geqq F(x)$ for all x and

$$\begin{aligned}\int_0^\infty & y^t \log^+ y|dF^*(y)| \\ &= 3^t\log 3[1-F(3+)]+\sum_{k=4}^{\infty} k^t \log k[F(k-1+)-F(k+)] \\ &\leqq C+\sum_{k=3}^{\infty}\left[\left(\frac{k+1}{k}\right)^t\frac{\log(k+1)}{\log k}\right]k^t\log k[F(k+)-F(k+1+)] \\ &\leqq C+\left[\left(\frac{4}{3}\right)^t\frac{\log 4}{\log 3}\right]\sum_{k=3}^{\infty}k^t\log k[F(k+)-F(k+1+)] \\ &\leqq C+C\int_0^\infty y^t\log^+ y|dF(y)| < \infty.\end{aligned}$$

Let G^* correspond to F and H^* correspond to F^*, using (3.3) to define G^* and H^*. Since $F^*(x)\geqq F(x)$ for all x we see that $H^*(x)\geqq G^*(x)$ for all x. Now

$$\int_0^\infty y^t\log^+ y|dF^*(y)| < \infty$$

implies $y^tF^*(y)\to 0$ as $y\to\infty$. Thus H^* (and also G^*) is finite and converges to zero as $y\to\infty$. We will show that $\int_0^\infty y^t|dH^*(y)|<\infty$ which, since

$$\int_0^\infty y^t|dG^*(y)| \leqq \int_0^\infty y^t|dH^*(y)|,$$

will be enough to guarantee that G^* satisfies (2.11) and, from Proposition 1 that $G^*\in\mathcal{G}$.

If $F^*(x)=0$ for $x\geqq N$ then $H^*(x)=0$ for $x\geqq N$ so $\int_0^\infty y^t|dH^*(y)|<\infty$ and we are done.

Assume otherwise, i.e. that $F^*(x)>0$ for $0\leqq x<\infty$. Note that F^* is left continuous with all its points of decrease at discontinuities and with an infinite number of discontinuities all contained in the set $\{3, 4,\ldots\}$; that $y^tF^*(y)$ is left continuous, is strictly increasing in every interval $(N, N+1]$ for $N\geqq 3$, and has local maxima contained in the set $\{3, 4,\ldots\}$; and thus that there exist integers $3=x_0<x_1<x_2<\cdots$ such that

(a) $\sup_{y\geqq x} y^tF^*(y)=x_k^tF^*(x_k)$ for $x_{k-1}<x$ and $k=1, 2,\ldots$;

(b) $x_1^tF^*(x_1)>x_2^tF^*(x_2)>\cdots$;

(c)
$$\begin{aligned}H^*(x) &= \max\left\{1, \sup_{y\geqq 1} y^tF^*(y)\right\} && 0\leqq x\leqq 1\\ &= \frac{1}{x^t}\sup_{y\geqq 1} y^tF^*(y) && 1\leqq x\leqq 3\\ &= \frac{1}{x^t}x_k^tF^*(x_k) && x_{k-1}<x\leqq x_k \text{ and } k=1,2,\ldots;\end{aligned}$$

(d) $H^*(x_k)=F^*(x_k)$ for $k=1, 2, \ldots$.

Define $p_k=F^*(x_k)-F^*(x_{k+1})$ and note that $\sum_{k=1}^{\infty} x_k^t \log x_k p_k<\infty$. Now

$$\int_0^\infty y^t|dH^*(y)| = \int_{(0,x_1)} y^t|dH^*(y)|+\sum_{k=1}^{\infty} x_k^t|\Delta H^*(x_k)| + \sum_{k=2}^{\infty}\int_{(x_{k-1},x_k)} y^t|dH^*(y)|. \tag{3.4}$$

The first expression on the right-hand side of (3.4) is finite. The second expression is easily seen to be finite as follows:

$$\sum_{k=1}^{\infty} x_k^t|\Delta H^*(x_k)| = \sum_{k=1}^{\infty} x_k^t\left[F^*(x_k)-\left(\frac{x_{k+1}}{x_k}\right)^t F^*(x_{k+1})\right] \leqq \sum_{k=1}^{\infty} x_k^t p_k < \infty.$$

The third expression on the right-hand side of (3.4) is a little more difficult to deal with.

Define $\mathscr{K}=\{k \mid k$ is a positive integer and $F^*(x_k)/p_k\leqq 2\}=\{k_1<k_2<\cdots\}$. Then

$$\begin{aligned}\sum_{k\geqq 1;\, k\notin\mathscr{K}} \int_{(x_k,x_{k+1})} y^t|dH^*(y)| &\leqq \sum_{k\geqq 1;\, k\notin\mathscr{K}} x_{k+1}^t \int_{(x_k,x_{k+1})} |dH^*(y)| \\ &\leqq \sum_{k\geqq 1;\, k\notin\mathscr{K}} x_{k+1}^t \int_{[x_k,x_{k+1})} |dF^*(y)| \\ &\leqq \sum_{k\geqq 1;\, k\notin\mathscr{K}} \left(\frac{x_{k+1}}{x_k}\right)^t \int_{[x_k,x_{k+1})} x^t|dF^*(x)|.\end{aligned} \tag{3.5}$$

Since $x_{k+1}^t F^*(x_{k+1})<x_k^t F^*(x_k)$ for $k\geqq 1$ we see that

$$\left(\frac{x_{k+1}}{x_k}\right)^t < \frac{F^*(x_k)}{F^*(x_k)-p_k} = \frac{1}{1-[p_k/F^*(x_k)]} < 2$$

when $F^*(x_k)/p_k>2$ so that (3.5) is bounded by $2\int_0^\infty x^t|dF^*(x)|$ which is finite.

Also

$$\begin{aligned}\sum_{k\in\mathscr{K}} \int_{(x_k,x_{k+1})} y^t\,|dH^*(y)| &= \sum_{k\in\mathscr{K}} t x_{k+1}^t F^*(x_{k+1}) \int_{x_k}^{x_{k+1}} \frac{1}{y}\,dy \\ &\leqq \sum_{k\in\mathscr{K}} t x_{k+1}^t F^*(x_{k+1}) \log x_{k+1} \\ &= t\sum_i x_{k_i+1}^t \log x_{k_i+1}[p_{k_i+1}+\cdots+p_{k_{i+1}-1}+F^*(x_{k_{i+1}})] \\ &\leqq t\sum_i \left[\sum_{k=k_i+1}^{k_{i+1}-1} x_k^t \log x_k p_k + 2x_{k_{i+1}}^t \log x_{k_{i+1}} p_{k_{i+1}}\right] \\ &\leqq 2t\sum_{k=1}^{\infty} x_k^t \log x_k p_k < \infty.\end{aligned} \tag{3.6}$$

Since both (3.5) and (3.6) are finite, we have finished showing that (3.4) is finite and thus completed the proof of the main part of the theorem.

The following example shows that the converse is false. Let

$$\begin{aligned} F(x) &= 1 & 0 \leqq x \leqq 2 \\ &= C\int_x^\infty y^{-t-1}(\log y)^{-2}\,dy & x \geqq 2 \end{aligned}$$

where $t>0$ and $C=\left[\int_2^\infty x^{-t-1}(\log x)^{-2}\,dx\right]^{-1}$. In this case $x^tF(x)$ is strictly decreasing for $x \geqq 2$ from which it follows that $G^*=F_2$. Then

$$\int_0^\infty y^t \log^+ y|dF(y)| = \int_2^\infty \frac{1}{y\log y}\,dy \doteq \infty$$

but $\lim_{y\to\infty} G^*(y)=0$ and

$$\int_0^\infty y^t|dG^*(y)| = \int_1^\infty \frac{1}{y(\log y)^2}\,dy + C < \infty.$$

In this case $G^* \in \mathscr{G}$ but $\int_0^\infty y^t \log^+ y|dF(y)|$ is not finite.

Proof of Theorem 7. Fix $\alpha>0$, $t>0$ and choose $3<x_1<x_2<\cdots$ such that

(1) $g(x_k) \geqq k^\alpha$ for $k=1, 2, \ldots$;

(2) $\log x_{k+1} \geqq 2\log x_k$ for $k=1, 2, \ldots$.

Define

$$p_k = \mu g(x_k)/k^{1+\alpha}\max\{x_k^t \log x_k, g(x_k)\}$$

where μ is such that $\sum_{k=1}^\infty p_k=1$, and define $F(x)=\sum_{\{k:x_k\geqq x\}} p_k$. F can be thought of as the "tail" of the distribution of a random variable X such that $P\{X=\pm x_k\} =p_k/2$. Obviously $F(x)\to 0$ as $x\to\infty$.

$$\begin{aligned} \int_0^\infty \frac{x^t \log^+ x}{g(x)}|dF(x)| &= \sum_{k=1}^\infty \frac{x_k^t \log x_k}{g(x_k)}\,p_k \\ &= \mu\sum_{k=1}^\infty \frac{x_k^t \log x_k}{k^{1+\alpha}\max\{x_k^t \log x_k, g(x_k)\}} \\ &\leqq \mu\sum_{k=1}^\infty \frac{1}{k^{1+\alpha}} < \infty. \end{aligned}$$

However

$$\begin{aligned} \int_0^\infty x^t|dG^*(x)| &\geqq \sum_{k=1}^\infty \int_{(x_k,x_{k+1})} x^t|dG^*(x)| \\ &= \sum_{k=1}^\infty \int_{(x_k,x_{k+1})} \frac{t}{x}\left[\sup_{y\geqq x_{k+1}} y^tF(y)\right]dx \\ &\geqq \sum_{k=1}^\infty t x_{k+1}^t p_{k+1}\log\frac{x_{k+1}}{x_k} \end{aligned}$$

$$\geqq \frac{t}{2} \sum_{k=1}^{\infty} x_{k+1}^{t} \log x_{k+1} p_{k+1}$$

$$= C \sum_{k=2}^{\infty} \frac{x_k^t g(x_k) \log x_k}{k^{1+\alpha} \max \{x_k^t \log x_k, g(x_k)\}}$$

$$= C \sum_{k=1}^{\infty} \frac{\min \{x_k^t \log x_k, g(x_k)\}}{k^{1+\alpha}}$$

$$= \infty.$$

Thus G^* does not satisfy (2.11) and from Proposition 1 it follows that $\mathcal{G}$ is empty.

LEMMA. *If $A_1, \ldots, A_M$ are measurable subsets of a probability space, then*

$$P\left(\bigcup_{k=1}^{M} A_k\right) \geqq \sum_{k=1}^{M} P(A_k) - \sum_{1 \leqq j < k \leqq M} P(A_j \cap A_k). \tag{3.7}$$

Proof. Is simple using induction.

Proof of Theorem 8. Define

$$\begin{aligned} F_0(x) &= 1 && x = 0 \\ &= F(2-) && 0 < x \leqq 1 \\ &= F(2^{N+1}-) && 2^{N-1} < x \leqq 2^N \text{ and } N = 1, 2, \ldots, \end{aligned}$$

and note that F_0 is left continuous. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables which are symmetrically distributed about zero and satisfy $P\{|X_k| \geqq x\} = F_0(x)$ for $x \geqq 0$. Define

$$\begin{aligned} a_{N,k} &= a_N && 1 \leqq k \leqq \nu_N \\ &= 0 && k > \nu_N \end{aligned}$$

where $\{a_N\}$ and $\{\nu_N\}$ will be obtained later. Define

$$S_{N,k} = S_N - a_N X_k \quad \text{for } 1 \leqq k \leqq \nu_N$$

and

$$A_{N,k} = \{S_{N,k} \geqq 0, a_N X_k \geqq 1\} \qquad 1 \leqq k \leqq \nu_N.$$

Then from the preceding lemma

$$\begin{aligned} P\{|S_N| \geqq 1\} &\geqq P\left(\bigcup_{k=1}^{\nu_N} A_{N,k}\right) \\ &\geqq \sum_{k=1}^{\nu_N} P(A_{N,k}) - \sum_{1 \leqq j < k \leqq \nu_N} P(A_{N,j} \cap A_{N,k}) \\ &\geqq \frac{\nu_N}{4} F_0\left(\frac{1}{a_N}\right) - \frac{\nu_N^2}{8} \left[F_0\left(\frac{1}{a_N}\right)\right]^2 \\ &\geqq \frac{\nu_N}{4} F_0\left(\frac{1}{a_N}\right)\left[1 - \nu_N F_0\left(\frac{1}{a_N}\right)\right]. \end{aligned} \tag{3.8}$$

If $\limsup_{x\to\infty} x^t F(x) \neq 0$, choose $\{a_N\}$ such that $0 < a_N \leqq N^{-2\rho/t}$ and such that

$$\inf_N \left(\frac{1}{a_N}\right)^t F_0\left(\frac{1}{a_N}\right) > 0,$$

and choose

$$\nu_N = \min\left\{[N^{-\rho}a_N^{-t}],\ \left[1/2F_0\left(\frac{1}{a_N}\right)\right]\right\} \tag{3.9}$$

where $[y]$ is the integral part of y. Then

$$\frac{\nu_N}{4}F_0\left(\frac{1}{a_N}\right)\left[1-\nu_N F_0\left(\frac{1}{a_N}\right)\right] \geqq \frac{1}{8}\nu_N F_0\left(\frac{1}{a_N}\right). \tag{3.10}$$

Since $F_0(1/a_N) \to 0$ as $N \to \infty$, it follows that $[1/2F_0(1/a_N)] \to \infty$ and

$$[1/2F_0(1/a_N)]F_0(1/a_N) \to \tfrac{1}{2} \quad \text{as } N \to \infty.$$

Also $[N^{-\rho}a_N^{-t}] \geqq [N^\rho]$ and

$$[N^{-\rho}a_N^{-t}]F_0\left(\frac{1}{a_N}\right) = \{a_N^t[N^{-\rho}a_N^{-t}]\}\left(\frac{1}{a_N}\right)^t F_0\left(\frac{1}{a_N}\right) \geqq CN^{-\rho}.$$

Thus from (3.8)–(3.10) and the preceding discussion of the asymptotic behavior of $F_0(1/a_N)$ times each of the possible expressions for ν_N, it follows that for large enough N, $P\{|S_N| \geqq 1\} \geqq CN^{-\rho}$ and

$$\sum_{N=1}^{\infty} N^{\rho-1}P\{|S_N| \geqq 1\} \geqq \sum_{N=C}^{\infty} CN^{-1} = \infty.$$

Now suppose $\limsup_{x\to\infty} x^t F(x) = 0$. Then if we choose $\{\nu_N\}$ and $\{a_N\}$ so that $\nu_N a_N^t \leqq N^{-\rho}$ we will have

$$\nu_N F_0\left(\frac{1}{a_N}\right) = (\nu_N a_N^t)\left\{\left(\frac{1}{a_N}\right)^t F_0\left(\frac{1}{a_N}\right)\right\}$$

which converges to zero as $N \to \infty$ so that from (3.8), for sufficiently large N

$$P\{|S_N| \geqq 1\} \geqq \tfrac{1}{8}\nu_N F_0(1/a_N). \tag{3.11}$$

Since $x^t F(x) \to 0$ as $N \to \infty$ we see that $G^*(x)$ is finite for all x. Since $F_0(x) \leqq F(x)$ we see that if G_0 corresponds to F_0, then $G_0(x)$ is also finite for all x. In addition, for $x > 4$ we have $F(x) \leqq F_0(x/4)$ so that $G^*(x) \leqq G_0(x/4)$. Thus $\int_0^\infty x^t|dG^*(x)| = \infty$ implies $\int_0^\infty x^t|dG_0(x)| = \infty$. Let $x_1 < x_2 < \cdots$ be those points (denumerably many of them) for which $x_1 > 4$ and $F_0(x_i) = G_0(x_i)$ for all i. Note that $x_i = 2^k$ for some positive integer k and thus that $x_{i+1} \geqq 2x_i$ for all i. Define $p_k = F_0(x_k) - F_0(x_{k+1})$ and note that for $x_k < x \leqq x_{k+1}$ we have $G_0(x) = (1/x^t)x_{k+1}^t F_0(x_{k+1})$. Then

$$\begin{aligned}\int_0^\infty x^t|dG_0(x)| &= C + \sum_{k=1}^\infty \int_{(x_k, x_{k+1})} x^t|dG_0(x)| \\ &\quad + \sum_{k=1}^\infty x_k^t\left[F_0(x_k) - \left(\frac{x_{k+1}}{x_k}\right)^t F_0(x_{k+1})\right] \\ &\leqq C + \sum_{k=1}^\infty \int_{x_k}^{x_{k+1}} \frac{t}{x} x_{k+1}^t F_0(x_{k+1})\, dx + \sum_{k=1}^\infty x_k^t p_k \\ &= C + t\sum_{k=2}^\infty x_k^t F_0(x_k) \log\left(\frac{x_k}{x_{k-1}}\right) + \sum_{k=1}^\infty x_k^t p_k.\end{aligned}$$

Recall that $x_k \geqq 2x_{k-1}$ so that

$$\sum_{k=2}^\infty x_k^t F_0(x_k) \log\left(\frac{x_k}{x_{k-1}}\right) \geqq C \sum_{k=2}^\infty x_k^t F_0(x_k) \geqq C \sum_{k=2}^\infty x_k^t p_k.$$

Thus $\int_0^\infty x^t|dG_0(x)| = \infty$ implies

$$\sum_{k=2}^\infty x_k^t F_0(x_k) \log\left(\frac{x_k}{x_{k-1}}\right) = \infty.$$

Now let

$$\begin{aligned} N_k &= [x_k^{t/\rho}] \quad \text{for } k \geqq 1 \text{ and } N_0 = 0, \\ a_N &= 1/x_k \quad \text{for } N_{k-1} < N \leqq N_k, \end{aligned}$$

and

$$\nu_N = [N^{-\rho} a_N^{-t}] \quad \text{for all } N.$$

Note that $\nu_{N_k} = 1$ for all k and $\nu_N \geqq 1$ for all other N, also that $\nu_N a_N^t \leqq N^{-\rho}$. We thus have from (3.11)

$$\begin{aligned}\sum_{N=1}^\infty N^{\rho-1}P\{|S_N| \geqq 1\} &\geqq C\sum_{N=1}^\infty N^{\rho-1}\nu_N F_0\left(\frac{1}{a_N}\right) \\ &= C\sum_{k=1}^\infty F_0(x_k) \sum_{N_{k-1}<N\leqq N_k} N^{\rho-1}\nu_N \\ &\geqq C\sum_{k=1}^\infty F_0(x_k)x_k^t \sum_{N=N_{k-1}+1}^{N_k} N^{-1} \\ &\geqq C\sum_{k=1}^\infty x_k^t F_0(x_k) \log\frac{N_k}{N_{k-1}} \\ &\geqq C\sum_{k=C}^\infty x_k^t F_0(x_k) \log\left(\frac{x_k}{x_{k-1}}\right) = \infty.\end{aligned}$$

Proof of Theorem 9 for $t>2$. Let $\{y_k\}$ be an increasing sequence of positive real numbers such that $g(y_k) \geqq 2^k$ for all k, such that $y_k \geqq 2^k$, and such that either

(1) $g(y_k) \geqq y_k^t \log y_k$ for all k, or

(2) $g(y_k) < y_k^t \log y_k$ for all k.

Case 1. Suppose $g(y_k) \geqq y_k^t \log y_k$ for all k. Fix $\alpha>0$ and define

$$p_k = \mu / y_k k^{1+\alpha} \quad \text{for } k = 1, 2, \ldots$$

where μ is such that $\sum_{k=1}^{\infty} p_k = 1$, let

$$F(x) = \sum_{\{k: y_k \geqq x\}} p_k \quad \text{for } x \geqq 0,$$

and suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables such that $P\{X_n = \pm y_k\} = p_k/2$ for $k=1,2,\ldots$. Note that $E|X_n| < \infty$, $EX_n = 0$, and

$$\int_0^\infty \frac{x^t \log^+ x}{g(x)} |dF(x)| = \sum_{k=1}^\infty \frac{y_k^t \log y_k}{g(y_k)} p_k \leqq \sum_{k=1}^\infty p_k = 1.$$

Using some of the notation of the first part of the proof of Theorem 8, let $\nu_n \equiv 1$ and $a_{N,1} = N^{-\rho/t}$. Then

$$\sum_{k=1}^\infty |a_{N,k}|^t = N^{-\rho}, \qquad \sum_{k=1}^\infty |a_{N,k}|^2 = N^{-2\rho/t},$$

and

$$\begin{aligned} \sum_{N=1}^\infty N^{\rho-1} P\{|S_N| \geqq 1\} &= \sum_{N=1}^\infty N^{\rho-1} P\{|X_1| \geqq N^{\rho/t}\} \\ &\geqq CE|X_1|^t = C \sum_{k=1}^\infty \frac{y_k^{t-1}}{k^{1+\alpha}} \\ &\geqq C \sum_{k=1}^\infty \frac{2^{k(t-1)}}{k^{1+\alpha}} = \infty. \end{aligned}$$

Case 2. Suppose $g(y_k) < y_k^t \log y_k$ for all k. Let $0 < \varepsilon < 1/t$ and define $\lambda_1 = (t - 2t\varepsilon)/(t-2)$ and $\lambda_2 = (t - t\varepsilon)/(t-2)$. Note that $1 < \lambda_1 < \lambda_2$. We want a sequence $\{x_n\}$ of positive real numbers such that

(1) $\{x_n\}$ contains an infinite number of members of $\{y_k\}$,

(2) $\lambda_1 \leqq \log x_{n+1} / \log x_n \leqq \lambda_2$ for $n=1,2,\ldots$.

Let $x_0 = y_{i_0}$ for some $y_{i_0} > 1$. Choose $\nu \geqq 1$ such that $\lambda_1^\nu < \lambda_2^{\nu-1}$ and choose $y_{i_1} \geqq x_0^{\lambda_2^{\nu-1}}$. There exists $m_1 \geqq \nu$ such that $x_0^{\lambda_1^{m_1}} < x_0^{\lambda_2^{m_1-1}} \leqq y_{i_1} < x_0^{\lambda_2^{m_1}}$. Because $x_0^{\lambda^{m_1}}$ is continuous in λ, a value of λ, say λ_1^*, exists such that $\lambda_1 < \lambda_1^* < \lambda_2$ and $y_{i_1} = x_0^{(\lambda_1^*)^{m_1}}$. Define $n_0 = 0$, $n_1 = m_1$, and $x_k = x_0^{(\lambda_1^*)^k}$ for $1 \leqq k \leqq m_1$. Now starting with x_{n_1} instead of x_0 we repeat this procedure to obtain $y_{i_2} \geqq x_{n_1}^{\lambda_2^{\nu-1}}$, an integer $m_2 \geqq \nu$ such that $x_{n_1}^{\lambda_1^{m_2}} < x_{n_1}^{\lambda_2^{m_2-1}} \leqq y_{i_2} < x_{n_1}^{\lambda_2^{m_2}}$, a λ_2^* such that $\lambda_1 < \lambda_2^* < \lambda_2$ and $y_{i_2} = x_{n_1}^{(\lambda_2^*)^{m_2}}$; we define $x_k = x_{n_1}^{(\lambda_2^*)^{k-n_1}}$ for $n_1 + 1 \leqq k \leqq n_1 + m_2 = n_2$. Performing the obvious induction we obtain the desired sequence $\{x_k\}$ and an increasing sequence $\{n_k\}$ such that x_{n_k} is a member of $\{y_i\}$ for each k. Let $\Gamma=\{n_1, n_2, \ldots\}$, let $\beta>0$, and define

$$p_n = \frac{\alpha \min\{g(x_n), \log x_n\}}{n^{1+\beta}x_n^t \log x_n} \qquad n \notin \Gamma,$$

$$= \frac{\alpha}{x_n^t \log x_n} \qquad n \in \Gamma,$$

where α is such that $\sum_{n=1}^{\infty} p_n=1$. Define $F(x)=\sum_{\{k:x_k\geqq x\}} p_k$ for $x\geqq 0$ and suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables such that $P\{X_n=\pm x_k\}=p_k/2$ for $k=1, 2, \ldots$. Note that

$$E|X_k|^t = \alpha \sum_{n\notin\Gamma} \frac{\min\{g(x_n), \log x_n\}}{n^{1+\beta}\log x_n} + \alpha \sum_{n\in\Gamma} \frac{1}{\log x_n}$$

$$\leqq \alpha \sum_{n=1}^{\infty} \frac{1}{n^{1+\beta}} + \alpha \sum_{n=1}^{\infty} \frac{1}{(\log x_1)\lambda_1^{n-1}} < \infty$$

and that

$$E\frac{|X_k|^t \log^+|X_k|}{g(|X_k|)} = \alpha \sum_{n\notin\Gamma} \frac{\min\{g(x_n), \log x_n\}}{n^{1+\beta}g(x_n)} + \alpha \sum_{n\in\Gamma} \frac{1}{g(x_n)}$$

$$\leqq \alpha \sum_{n=1}^{\infty} \frac{1}{n^{1+\beta}} + \alpha \sum_{k=1}^{\infty} \frac{1}{g(x_{n_k})}$$

$$\leqq C + \alpha \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

Using the notation of the first part of the proof of Theorem 8 we set

$$N_0 = 0 \quad \text{and} \quad N_k = [x_k^{t/\rho}] \quad \text{for } k = 1, 2, \ldots$$

$$a_N = 1/x_k \quad \text{for } N_{k-1} < N \leqq N_k \text{ and } k = 1, 2, \ldots$$

$$\nu_N = [N^{-\rho}a_N^{-t}]$$

and note that $\sum_{k=1}^{\infty} |a_{N,k}|^t=\nu_N a_N^t \leqq N^{-\rho}$,

$$\sum_{k=1}^{\infty} |a_{N,k}|^2 = \nu_N a_N^2 \leqq N^{-\rho\varepsilon} N_{k-1}^{-\rho(1-\varepsilon)} x_k^{t-2}$$

$$\leqq CN^{-\rho\varepsilon} x_{k-1}^{-t(1-\varepsilon)} x_k^{t-2}$$

$$= CN^{-\rho\varepsilon} \exp\left\{(t-2)\log x_{k-1}\left[-\frac{t(1-\varepsilon)}{t-2}+\frac{\log x_k}{\log x_{k-1}}\right]\right\}$$

$$\leqq CN^{-\rho\varepsilon}$$

since by construction of the x_k's the exponent of e above is negative; we define $\gamma=\rho\varepsilon$. Now $E|X_k|^t<\infty$ implies $x^t F(x)\to 0$ as $x\to\infty$ so referring to the arguments of (3.7), (3.8), and (3.10) we have for large enough N

$$P\{|S_N| \geqq 1\} \geqq \tfrac{1}{8}\nu_N F(1/a_N)$$

so that

$$\begin{aligned}\sum_{N=1}^{\infty} N^{\rho-1}P\{|S_N| \geqq 1\} &\geqq C \sum_{N=C}^{\infty} N^{\rho-1}\nu_N F\left(\frac{1}{a_N}\right)\\ &\geqq C \sum_{k=C}^{\infty} x_k^t F(x_k) \sum_{N=N_{k-1}+1}^{N_k} N^{-1}\\ &\geqq C \sum_{k=C}^{\infty} x_k^t F(x_k) \log \frac{N_k}{N_{k-1}}\\ &\geqq C \sum_{k=C}^{\infty} x_k^t p_k \log \frac{x_k}{x_{k-1}}.\end{aligned}$$

Now $x_{k-1} \leqq x_k^{1/\lambda_1}$ so that $x_k/x_{k-1} \geqq x_k^{1-1/\lambda_1}$ and thus

$$\begin{aligned}\sum_{N=1}^{\infty} N^{\rho-1}P\{|S_N| \geqq 1\} &\geqq C \sum_{k=C}^{\infty} x_k^t p_k \log x_k\\ &\geqq C \sum_{k=C}^{\infty} x_{n_k}^t p_{n_k} \log x_{n_k}\\ &= C \sum_{k=C}^{\infty} \alpha = \infty.\end{aligned}$$

4. **Concluding remarks.** Results for $t<1$ and results of the sharpness of Theorems 1, 2, 3, and 5 are of some interest though investigations of these matters should be fairly routine.

For $t \leqq 2$ we see that

$$\sum_{k=1}^{\infty} |a_{N,k}|^2 \leqq \left[\max_k |a_{N,k}|\right]^{2-t} \sum_{k=1}^{\infty} |a_{N,k}|^t$$

so that $\gamma_N \to 0$ at least as fast as ρ_N does. So for $t \leqq 2$ no assumption need be made on the rate at which $\gamma_N \to 0$. (In Theorems 4 and 5 the assumption $\gamma > 0$ is automatically satisfied if $t \leqq 2$.) Suppose $t > 2$, the X_k's are independent random variables normally distributed with common mean zero and common variance one, and $\sum_{k=1}^{\infty} |a_{N,k}|^2 = 1$ for all N. Then S_N is itself normally distributed with mean zero and variance one for all N so that $P\{|S_N| > \varepsilon\}$ is a nonzero constant so the results of our theorems cannot hold no matter how fast $\rho_N \to 0$. The assumptions of Theorems 1c, 2c, 3, 4, and 5 all relate the rates at which $\rho_N \to 0$ and $\gamma_N \to 0$. The above argument shows that $\gamma_N \to 0$ is necessary because of considerations related to the Central Limit Theorem. Is it actually necessary that the rates at which $\rho_N \to 0$ and $\gamma_N \to 0$ be related, and if so what is the minimum restriction that need be put on the γ_N's?

It would be very nice to combine the assumptions on the moments with those on the coefficients in some way, and then to generalize this type of result so as to deal with probabilities of the form

$$P\left\{\int f(t)\, d[X_t - EX_t] \geqq \varepsilon\right\}$$

where $\{X_t\}$ is a continuous parameter process with independent increments. This sort of thing was done in a different situation in [7] and [8] but would appear to be harder in this case.

Also of interest are both exponential and algebraic rates of convergence for quadratic (and higher order) sums of independent random variables. At least in the quadratic case this convergence is of some practical interest. Unfortunately, obtaining results in these cases seems to be very hard and to require new techniques. The authors have been able to obtain only fragmentary results of this type.

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